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Non-Classical Symmetry Solutions to the Fitzhugh Nagumo Equation.

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ABSTRACT

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In *Reaction-Diffusion* systems, some parameters can influence the behavior of other parameters in that system. Thus reaction diffusion equations are often used to model the behavior of biological phenomena. The Fitzhugh Nagumo partial differential equation is a reaction diffusion equation that arises both in population genetics and in modeling the transmission of action potentials in the nervous system. In this paper we are interested in finding solutions to this equation. Using Lie groups in particular, we would like to find symmetries of the Fitzhugh Nagumo equation that reduce this non-linear PDE to an Ordinary Differential Equation. In order to accomplish this task, the non-classical method is utilized to find the *infinitesimal generator* and the *invariant surface condition* for the subgroup where the solutions for the desired PDE exist. Using the infinitesimal generator and the invariant surface condition, we reduce the PDE to a mildly nonlinear ordinary differential equation that could be explored numerically or perhaps solved in closed form.

To my Brother,

Siavash Mehraban,

who is no longer with us.

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CONTENTS

1 FINDING SOLUTIONS TO DIFFERENTIAL EQUATIONS

1.1 Differential Equations

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things change. When expressed in mathematical terms, the relations are equations and the rates are derivatives [1]. Equations containing derivatives are ***differential equations***. A differential equation that describes some physical process is often called a ***mathematical model*** of the process [1] .

Differential equations are of interest to non-mathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable one to make predictions about how the natural process will behave in various circumstances.

Differential Equations can be classified in different ways. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables [1] . In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an ***ordinary differential equation*** (ODE). In the second case, the derivatives are partial derivatives, and the equation is called a ***partial differential equation*** (PDE).

The set of equations (1) are examples of ordinary differential equations known as

Lotka-Volterra [1], or predator-prey, equations

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= -cy + \gamma xy \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species and the constants of a, α, c, and γ are based on empirical observations and depend on the particular species being studies.

On the other hand, equation (2) is an example of a partial differential equation known as the heat equation that describes how heat energy spreads out if the heat energy is initially concentrated in one place, where k is a constant [5]:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2}$$

The ***order*** of a differential equation is the order of the highest derivative that appears in the equation. For example, equations (1) are both first order ordinary differential equations, whereas, (2) is a second order partial differential equation.

A crucial classification of the differential equations is whether they are linear or nonlinear [1] . The ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is said to be ***linear*** if F is a linear function of the variables, $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus, the general linear ordinary differential equation of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_n(t)\, y = g(t) \tag{3}$$

An equation that is not of the form (3) is ***nonlinear***. Equation (4) is an example of a nonlinear equation known as ***Fitzhugh Nagumo*** equation

$$u_t = u_{xx} - u^3 + (a+1)u^2 - au \tag{4}$$

[2].

1.2 Reaction Diffusion

As mentioned earlier, if the heat energy is initially concentrated in one place, equation (2) describes how the heat energy spreads out, a physical process known as ***diffusion*** [5]. Other physical quantities besides temperature smooth out in much the same manner, satisfying (2) [5]. For this reason, (2) is also known as the ***diffusion equation***. The diffusion equation can be applied in biomedical sciences. For example the behavior of *axon* in a cell which is a component along which output electrical signals propagate [6] is treated as a diffusion equation. The output signal is called an ***action potential*** [6]. Electric signaling or firing by individual nerve cells or neurons is particularly common [6]. The seminal and now classical work by ***Hodgkin and Huxley*** on this aspect of nerve membranes was on the nerve axon of the giant squid.

Basically the axon is a long cylindrical tube which extends from each neuron and electrical signals propagate along its outer membrane, about 50 to 70 Angstrom thick [4]. Electrical pulses arise because the membrane is preferentially permeable to various chemical ions with the permeabilities affected by the currents and potentials present. The deviation in the potential across the membrane, measured from resting state, is observable in experiments. The membrane permeability properties change when subjected to a stimulating electrical current I; they also depend on potential.

Such a current can be generated, for example, by local depolarization relative to the rest state.[4]

Hodgkin and Huxley formulated a system of equations

$$C_m \frac{\partial V}{\partial t} = \frac{a}{2\rho_i} \frac{\partial^2 V}{\partial x^2} + g_k n^4 (V_k - V) + g_{Na} m^3 h (V_{Na} - V) + g_l (V_l - V) + I_A \qquad (5)$$

$$\frac{\partial n}{\partial t} = \alpha_n (1 - n) - \beta_n n \qquad (6)$$

$$\frac{\partial m}{\partial t} = \alpha_m (1 - m) - \beta_m m \qquad (7)$$

$$\frac{\partial h}{\partial t} = \alpha_h (1 - h) - \beta_h h \qquad (8)$$

wherein C_m is the membrane capacitance per unit area, ρ_i is the intracellular resistivity, g_k, g_{Na}, g_l, and $I_A$are conductance and applied current density per unit area [4].

This system is often called the *full or complete Hodgkin-Huxley system* [4]. The second space derivative in (5) enables the depolarization at one set of space points to initiate changes at neighboring space points. The possibility arises of a local response (solutions of an equation known as the cable equation), but there is also the possibility of propagating action potentials [6].

The Hodgkin-Huxley equations (5)-(8) are in the form of a ***reaction diffusion system***. An example of the form of a such system when there is just one dependent variable $u(x,t)$ is

$$u_t = Du_{xx} + F(u) \qquad (9)$$

where $D > 0$ is the diffusion coefficient (in square distance per time). The quantity u may represent the concentration of a chemical, which, in the absence of other effects, diffuses according to the *diffusion (heat)* equation

$$u_t = Du_{xx} \tag{10}$$

The terminology for reaction-diffusion equations comes from the chemical literature [4].

1.3 Fitzhugh Nagumo Equation

The analysis of the Hodgkin-Huxley equations (5)-(8) is extremely difficult because of the nonlinearities and the large number of variables [4]. Although there are efficient numerical methods for some forms of these systems of equations, it is a formidable task to compute solutions with all different sets of parameters, different applied currents, and different boundary conditions of interest [4]. Mathematical analysis would be helpful even if it were performed on simpler equations whose solutions shared the qualitative properties of those of the Hodgkin-Huxley equations. Analysis of such simpler systems may lead to the discovery of new phenomena, which may then be searched for in the original system and also in experimental preparations.

Such a simplified system of equations has its origins in the works of Fitzhugh (1961) and Nagumo, Arimoto, and Yoshizawa (1962) and has become known as the ***Fitzhugh-Nagumo equations*** [4]. In the Hodgkin-Huxley system the variable V (voltage) and m (sodium activation) have similar (mostly fast) time courses and the variables n (potassium activation) and h (sodium inactivation) have similar (slower) time courses. Heuristically speaking, in the Fitzhugh-Nagumo systems, V and m are regarded as mimicked by a single variable $v(x, t)$ which we will call voltage, and a and h are mimicked by a single variable $w(x, t)$, which is called the *recovery variable*

[4]. The Fitzhugh-Nagumo equations in their general form are

$$v_t = v_{xx} + f(v) - w \tag{11}$$

$$w_t = b\,(v - \gamma w) \tag{12}$$

where $f(v)$ is the cubic

$$f\,(v) = v\,(1 - v)\,(v - a) \tag{13}$$

where $0 < a < 1$, and where a and γ are positive constants. A term $I = I(x, t)$ representing an applied current may be inserted on the right-hand side of (11) [4]. Often, γ is set equal to zero in which case we will refer to the *simplified Fitzhugh-Nagumo equations*

$$v_t = v_{xx} + f(v) - w \tag{14}$$

$$w_t = bv \tag{15}$$

which are sometimes combined into the single equation

$$v_t = v_{xx} + f\,(v) - b \int_0^t v\,(x, t')\,dt' \tag{16}$$

where the last term takes the form of a *killing* term. If we set $b = 0$, we obtain the *reduced Fitzhugh-Nagumo equation* with just one component

$$v_t = v_{xx} + f\,(v) \tag{17}$$

with kinetic (space clamped) equation

$$\frac{dv}{dt} = f\,(v) \tag{18}$$

[4]. It is this form of the Fitzhugh-Nagumo equations that we will be working with in the remainder of this thesis.

2 SYMMETRY ANALYSIS OF PDES

The Fitzhugh-Nagumo equation is a non-linear PDE which is still difficult to solve [7]. Numerical methods could be utilized to obtain approximate solutions of this equation, but we are interested in symmetry reduction of this equation to be able to find its exact analytical solutions. Symmetry group techniques provide one method for obtaining such solutions and, furthermore, they do not depend on whether or not the equation is integrable.

Symmetry groups and associated reductions and exact solutions have several different applications in the context of non-linear differential equations:

1. *Derive new solutions from old solutions.* Applying the symmetry group of a differential equation to a known solution yields a family of new solutions. Quite often, interesting solutions could be obtained from trivial ones.

2. *Integration of ODEs.* Symmetry groups of ODEs can be used to reduce the order of the equation, for example, to reduce a second-order equation to a first-order one.

3. *Reduction of PDE's.* Symmetry groups of PDE's are used to produce the total number of dependent and independent variables; for example, from a PDE with two independent and one dependent variable to an ODE.

4. *Linearization of PDE's.* Symmetry groups can be used to discover whether or not a PDE can be linearized and to construct an explicit linearization when one exists.

5. *Classification of equations.* Symmetry groups can be used to classify differential equations into equivalence classes and to choose simple representatives of such classes.

6. *Asymptotics of solutions of PDEs.* It is known that as solutions of PDEs asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be effectively used to study properties such as asymptotics and 'blow up'.

7. *Numerical methods and testing computer coding.* Symmetry groups and exact solutions of physically relevant PDEs are used in the design, testing, and evaluation of numerical algorithms.

8. *Conservation laws.* The application of symmetries to conservation laws dates back to the work of Noether, who proved the remarkable result that, for systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property.

9. *Further applications.* There are several other important applications of symmetry groups, including bifurcation theory, control theory, special function theory, boundary value problems and free boundary problems.

[7].

In the mid-nineteenth century, *Sophus Lie* was searching for a general theory for solving differential equations [7]. He made the profound and far-reaching discovery that many special methods for first order ODEs – such as separable equations, homogeneous equations, exact equations, and integrating factors – were, in fact, special cases of a general integration method based on the invariance of differential equations under a continuous group of symmetries, known as the *Lie group* [7]. Lie developed a theory of for symmetry groups of differential equations which is highly algorithmic [7]. This method is now known as classical Lie method for finding group-invariant solutions.

Subsequently, there have been many applications in numerous areas of mathematics, physics, chemistry, engineering, and elsewhere [7] . However, the method is often quite difficult because the task of finding the symmetry group of a given system of differential equations is often exceedingly cumbersome. Despite the fact that the method is entirely algorithmic, it usually involves a large amount of tedious algebra and the associated calculations can be virtually unmanageable if attempted manually [7].

In this paper we are interested in applying the Lie method to the Fitzhugh Nagumo equation to find its symmetry solutions and reduce it to an ordinary differential equation. The method of point symmetry analysis involves looking for a Lie group of invertible transformations that map every solution of the differential equation to another solution of the differential equation [9].

We elaborate on the concept of symmetry solutions of the Lie method by applying this method to much simpler ordinary differential equations (ODE) as an example. We start with solving an ODE by finding its integrating factor. Then, for the same ODE, we find the symmetry group that the ODE is invariant over. This is known as *classical method* [7]. Next, we compare the results of applying the classical method with integrating factor method to support the notion that the integrating factor is a special case of the general integration method based on the invariance of differential equations under a continuous group of symmetries as mentioned above. As the final step, we introduce the nonclassical method and apply it to Fitzhugh Nagumo equation to reduce it to an ODE that could be solved numerically or analytically.

Suppose we have a differential equation of the form

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} \tag{19}$$

to solve. For simplicity, we refer to $M(x,y)$ by M and $N(x,y)$ by N and rewrite equation (19) in the form

$$Mdx + Ndy = 0 \tag{20}$$

This is called the ***differential form*** of equation (19) [1].

Given any differential equation of the form (20), there exists a $\mu(x,y)$ such that

$$\mu Mdx + \mu Ndy = 0 \tag{21}$$

Equation of the form (21) is called the ***exact form*** for the differential form (20) and μ is called an integrating factor for it [1]. Exact means there exists an $\omega(x,y)$ such that

$$M = \frac{\partial \omega}{\partial x} \text{ and } N = \frac{\partial \omega}{\partial y}$$

For example, let us find solutions to the equation

$$2ydx + xdy = 0 \tag{22}$$

by finding an *integrating factor*. From equation (22) we have

$$\frac{\partial \omega}{\partial x} = 2y \tag{23}$$

$$\frac{\partial \omega}{\partial y} = x \tag{24}$$

Equations (23) and (24) result in

$$\frac{\partial^2 \omega}{\partial x \partial y} = 2$$

$$\frac{\partial^2 \omega}{\partial y \partial x} = 1$$

so this equation is not exact.

However, if we multiply both sides of equation (22) by an x, we have

$$x(2ydx + xdy) = 0$$

or equivalently

$$2xydx + x^2dy = 0$$

It then follows that

$$\omega_x = 2xy$$

$$\omega_y = x^2$$

which results in

$$\omega_{xy} = 2x \quad \text{and} \quad \omega_{xy} = 2x$$

So if $\omega = k$ then $x^2y = k$ and

$$y = \frac{k}{x^2} \tag{25}$$

We can verify the results in (25) by finding solutions to (22) with the separation by parts method:

$$2ydx + xdy = 0$$

$$\frac{dy}{y} = \frac{-2dx}{x}$$

$$\ln(y) = -2\ln(x) + C$$

Therefore

$$y = \frac{k}{x^2} \tag{26}$$

So $\omega(x, y) = x$ is an integrating factor for $2ydx + xdy = 0$.

It follows that $Mdx + Ndy = 0$ is exact if $M_y = N_x$. It also follows that the integrating factor $\mu(x, y)$ satisfies

$$\mu M dx + \mu N dy = 0$$

where

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial y}(\mu N)$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \tag{27}$$

Finding solutions to equation of the form (27) is in general more difficult than finding solutions to equations of the form (19).

Thus, in order to find solutions to first order differential equations, we often use *invariance over a group* for the differential equation to determine an integrating factor for it. For that purpose we start with definition of a ***Group*** [3].

Definition 3.1 *[3]A **Group** is a set G equipped with a binary operation $*$ such that*

1) for every $x, y, z \in G : x * (y * z) = (x * y) * z$

2) there is an element $e \in G$, the identity, with $e * x = x = x * e$ for all $x \in G$

3) every $x \in G$ has an inverse, there is a $x^{'} \in G$ with $x * x^{'} = e = x^{'} * x$

The properties in the definition of a group depict the shared properties of two nonempty sets in general, furnished with a binary operator. Such shared properties could provide a model describing how elements in a group look like. For example, a square can be rotated through a certain angle and then reflected about a line, and the square will end up with its original shape. Thus, the rotation and reflection are the symmetries of the square that will not cause a change in form for a square. Groups could be discrete or continuous. We are interested in finding such symmetries for a differential equation. We would like to know if certain symmetries applied to the solutions of a differential equation, how will the solution differ? In order to do that, we need to limit the desired groups to certain groups known as Lie groups which are continuous groups. The solutions for a differential equation could be a set that depicts a line, a surface or a sphere. *Lines* and *circles* are ***one-dimensional manifolds*** and *planes* and *spheres* are ***two-dimensional manifolds*** [2]. In order to understand what a manifold is we need to know what a chart is.

Definition 3.2 *[2] Let M be a topological space. A local* ***chart*** *on M is a pair (U, Φ) where U is an open set in M and Φ is a homomorphism of U onto an open set $\Phi(U)$ in $\mathbb{R}^n$ for some n.*

The definition of the chart depicts that a transition that maps the compatibility of change from one set of coordinates to another set of coordinates is continuous. Having defined a chart we can move to a smooth manifold definition. A smooth manifold is defined as follows:

Definition 3.3 *[2] A* ***smooth manifold*** *M is a topological space together with a collection of local charts (Φ, U_α) (called an* ***atlas****) such that $\{U_\alpha\}$ cover M and for* *each pair, α, β the mapping $\Phi_\beta \circ \Phi_\alpha^{-1}$ is a C^∞ mapping from $\Phi_\alpha(U_\alpha \cap U_\beta)$ to $\Phi_\beta(U_\alpha \cup U_\beta)$.*

That is, if two charts overlap, the transition map between them is still smoothly compatible. This notion is very important in solving differential equations. This is why we are interested in the symmetry solutions of a differential equation. To solve differential equations, we need the combination of the concept of group with that of a differentiable manifold. This leads to the following:

Definition 3.4 *[2] A **Lie group** is an algebraic group that forms a manifold in which the multiplication is differentiable and each point in there has a tangent.*

For example, the set of all 2-dimensional rotations matrices R_θ through an angle θ forms the group

$$SO(2) = \left\{ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \quad | \quad \theta \in [-\pi, \pi] \right\}$$

where S means the determinant is 1 and the letter O indicates orthogonality.

4 CLASSICAL METHOD

4.1 Invariance Over a Group

To explain what invariance over a group for a differential equation means, we start with the equation

$$x'' = -x \tag{28}$$

as an example and show that invariance over certain groups does not change the properties of this equation. In order to do that we rewrite equation (28) as

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

or

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so that

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = R_{\pi} \begin{bmatrix} x \\ y \end{bmatrix}$$

Suppose we transform x and y to new positions u and v using the rotation matrix as follows

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where

$$G = \{R_{\theta} | \text{ such that } \theta \in [-\pi, \pi]\}$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = R_{-\theta} \begin{bmatrix} u \\ v \end{bmatrix} \tag{29}$$

because $(R_\theta)^{-1} = R_{-\theta}$.

We take the derivative of both sides of the equation (29)

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = R_{-\theta} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

Consequently, in terms of u and v variables we have

$$\begin{aligned} R_{-\theta} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} R_{-\theta} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} &= R_\theta R_\pi R_{-\theta} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= R_{\theta+\pi-\theta} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= R_\pi \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

Finally, we obtain

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = R_\pi \begin{bmatrix} u \\ v \end{bmatrix}$$

Therefore, we have shown that equation (28) is invariant over the rotation group, R_θ, and how a differential equation does not necessarily change form under a change of variable for a family of coordinate transformations, like a rotation group. If that is the case, the differential equation is called *invariant over G*.

4.2 Classical Method

In order to explain what the classical method for invariance over a group means, we revisit the equation $x'' = -x$ in the previous section. We showed that this equation is invariant over the rotation group $G = R_\theta$.

Note that,

$$
\begin{aligned}
\frac{dG}{d\theta} &= \begin{bmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{bmatrix} \\
\\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
\end{aligned}
$$

Thus, formally we have

$$G_\theta = e^{\Gamma\theta}$$

where

$$\Gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We call Γ the infinitesimal generator of G_θ [7].

Let us also define $e^{\Gamma\theta}$ formally by

$$\sum \frac{\Gamma^n \theta^n}{n!}$$

In terms of our example, we can rewrite $e^{\Gamma\theta}$ as

$$
\begin{aligned}
e^{\Gamma\theta} &= e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\theta} \\
&= I + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\theta + \begin{bmatrix} 0 & -1 \\ 1 & \theta \end{bmatrix}^2 \frac{\theta^2}{2!} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 \frac{\theta^3}{3!} + \dots \\
&= \begin{bmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots & 0 - \theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots \\ 0 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= R_\theta
\end{aligned}
$$

In general, if G_θ is a 1-parameter group, then

$$\Gamma = \left.\frac{dG}{d\epsilon}\right|_{\epsilon=0}$$

is the infinitesimal generator. Then

$$G_\epsilon G_\theta = G_{\epsilon+\theta}$$

Therefore,

$$\frac{dG_\epsilon}{d_\epsilon} G_\theta = \frac{dG_\epsilon}{d_{\epsilon\theta}} \frac{d_{\epsilon\theta}}{d_\epsilon}$$

Thus

$$\left.\frac{dG_\epsilon}{d_{\epsilon\theta}}\right|_{\epsilon=0} G_\theta = \left.\frac{d_{\epsilon\theta}}{d_{\epsilon+\theta}}\right|_{\epsilon=0} = 0$$

so

$$\Gamma G_\theta = \frac{dG_\theta}{d\theta}$$

It then follows that

$$G_\theta = e^{\Gamma\theta}$$

In order to determine if a differential equation,

$$Mdx + Ndy = 0$$

is invariant over a group

$$G_t(x, y),$$

we need to verify that the differential equation remains unchanged under transformations defined by such G_t. If we can verify the invariance over a group G_t, then the differential equation can be solved with the following steps:

1) Compute

$$\frac{d}{dt}G_t\bigg|_{t=0} = G_0' = \begin{bmatrix} P \\ Q \end{bmatrix}$$

2) Solve

$$\frac{dx}{dy} = \frac{P(x, y)}{Q(x, y)}$$

and denote the solution by $u(x, y)$. It can be shown that $u(x, y)$ is also invariant over the group.

3) Substitute $u(x, y)$ into the equation. The equation becomes separable.

This is known as the ***Classical Method*** for first order ordinary differential equations.

Example: Solve

$$\frac{dx}{dy} = \frac{x^2 + y^2}{xy} \tag{30}$$

where

$$G_t(x, y) = (e^t x, e^t y)^T$$

Solution:

1)

$$G_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^0 x \\ e^0 y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

So G_0 is identity because it maps (x, y) to (x, y). Also

$$\begin{aligned} G_s G_t \begin{bmatrix} x \\ y \end{bmatrix} &= G_s \begin{bmatrix} e^t x \\ e^t y \end{bmatrix} \\ &= \begin{bmatrix} e^s e^t x \\ e^s e^t y \end{bmatrix} \\ &= \begin{bmatrix} e^{s+t} x \\ e^{s+t} y \end{bmatrix} \\ &= G_{s+t} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

So if

$$G_t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t x \\ e^t y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Then

$$x_1 = e^t x \Rightarrow x = e^{-t} x_1$$

$$y_1 = e^t y \Rightarrow y = e^{-t} y_1$$

Therefore

$$\frac{dy}{dx} = \frac{e^t y_1}{e^t x_1} = \frac{dy_1}{dx_1}$$

So

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

could be translated to

$$\begin{aligned}\frac{dy_1}{dx_1} &= \frac{(e^{-t}x_1)^2 + (e^{-t}y_1)^2}{(e^{-t}x_1)^2(e^{-t}y_1)^2} \\ &= \frac{e^{-2t}(x_1^2 + y_1^2)}{e^{-2t}(x_1^2 y_1^2)} \\ &= \frac{x_1^2 + y_1^2}{x_1^2 y_1^2}\end{aligned}$$

2) Now we find

$$\frac{d}{dt}G_t\bigg|_{t=0} = G_0'$$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

We let $u = e^C$ and integrate both sides:

$$\begin{aligned}\ln(x) &= \ln(y) + C \\ y &= ux \\ u &= \frac{y}{x} \qquad (31)\end{aligned}$$

If we substitute the result (31) in the differential equation, we get a separable differential equation

$$\frac{dy}{dx} = \frac{du}{dx}x + u\frac{d}{dx}x$$

3) So we substitute the above in

$$\frac{dx}{dy} = \frac{x^2 + y^2}{xy}$$

Then

$$\begin{aligned}
\frac{x}{du}dx + u &= \frac{x^2 + u^2x^2}{x(ux)} \\
&= \frac{1+u^2}{u} \\
&= \frac{1}{u} + u
\end{aligned}$$

So

$$x\frac{du}{dx} = \frac{1}{u} \Rightarrow udu = \frac{dx}{x}$$

and

$$\frac{u^2}{2} = \ln(x) + C$$

and

$$u^2 = \frac{y^2}{x^2}$$

$$\frac{y^2}{2x^2} = \ln(x) + C$$

Therefore we showed how we could find solutions to a first order differential equation by using group invariance.

We would like to elaborate on the properties of such groups to be able to find solutions to more complex differential equations such as partial differential equations.

5 THE NON-CLASSICAL METHOD

Using the classical method to find solutions to ordinary differential equations is ideal in many cases. However, the classical method applied to partial differential equations (PDE) sometimes produces trivial solutions only. That is because the symmetry group of non-trivial solutions to the PDE is a subgroup of the group produced by the classical method (and the larger the group, the fewer its invariant solutions) [7].

To find the non-trivial solutions we would have to set up new conditions for the PDE. Before proceeding with this notion we need to introduce several definitions. We start with the point-transformation definition of a *1-parameter continuous group.* Here, we formally define the *1-parameter continuous group* as the solutions of PDE's exist in such groups.

Definition 5.1 *A set G of point transformations in the (x, y) plane $\mathbb{R}^2$,*

$$x^* = f(x, y, a) \tag{32}$$

$$y^* = g(x, y, a) \tag{33}$$

*depending on the parameter a is called a **one-parameter continuous group**, if*

- $x = f(x, y, 0)$ *and* $y = g(x, y, 0)$ *(identity).*
- $x = f(x^*, y^*, -a)$ *and* $y = g(x^*, y^*, -a)$ *(inverse).*
- *If* $x^* = f(x, y, a), y^* = g(x, y, a)$ *and* $x^{**} = f(x^*, y^*, b), y^{**} = g(x^*, y^*, b)$ *then*

$$x^{**} = f(x, y, a + b), y^{**} = g(x, y, a + b)$$

Given a one-parameter continuous group, we would like to know if the solutions to the desired differential equation remain invariant under such a group. That is, if the solutions of the desired PDE is transformed to new positions, the transformations (32) and (33) won't get changed. Thus, we formally define a symmetry group for a set of transformation in a continuous group.

Definition 5.2 *It is said that the transformations* (32) *and* (33) *form a* ***symmetry group*** *of a differential equations* $F(x, y, y_x, y_{xx}) = 0$ *if the equation is form invariant, i.e. if* $F(x, y, y_x, y_{xx}) = 0$*, then* $F(x^*, y^*, y^*_{x^*}, y^*_{x^{**}}) = 0$.

Example:

$$x''(t) = -x(t)$$

indicates

$$x''(t + s) = -x(t + s)$$

which is invariant under translation in time. So

$$x^*(t) = x(t + s)$$

$$y^*(t) = y(t + s)$$

which result in $G_s(x(t), y(t))$. Therefore

$$G_s(x(t), y(t)) = (x(t + s), y(t + s))$$

is a 1-parameter symmetry group.

These definitions allow us to generalize our earlier work with the classical method to any ordinary differential equation of the form $F(t, p, p_t, p_{tt}) = 0$. Lie's theory relates the topological component of the symmetry group G containing the identity to the infinitesimal transformations of its 1-parameter subgroups, where the infinitesimal generator of a 1-parameter group G_ϵ is the linear part of its Maclaurin series expansion in ϵ,

$$t^* = t + \epsilon T(t, p) + O(\epsilon^2)$$

$$p^* = p + \epsilon P(t, p) + O(\epsilon^2)$$

It is convenient to represent an infinitesimal transformation of the above form by a linear differential operator

$$\Gamma = T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p}$$

and call it the ***infinitesimal generator*** or ***group generator***.

In order to apply the theory to a differential equation, a Lie Group G which acts on a set of functions $\{x(t), p(t)\}$ that are solutions to a system of differential equations must be extended to include the derivatives in t of those functions. That is, the representation of the 1-parameter group G_ϵ on the solutions is extended (i.e., prolonged) to a representation $G_\epsilon^{(1)}$ acting on the solutions and their derivatives. The infinitesimal generator of this extended representation is known as the first prolongation of Γ, denoted by $\Gamma^{(1)}$. In general, suppose we have

$$(x^*, p^*) = (Q(p, u), \eta(p, u))$$

We apply G to p_x

$$
\begin{aligned}
p^*_{x^*} &= \frac{\dfrac{dp^*}{dx}}{\dfrac{dx^*}{dx}} \\
&= \frac{\dfrac{\partial \eta}{\partial x}\dfrac{\partial x}{\partial x} + \dfrac{\partial \eta}{\partial p}\dfrac{\partial p}{\partial p}}{\dfrac{\partial Q}{\partial x}\dfrac{\partial x}{\partial x} + \dfrac{\partial Q}{\partial p}\dfrac{\partial p}{\partial x}} \\
&= \frac{\dfrac{\partial}{\partial x}\eta + p_x\dfrac{\partial \eta}{\partial p}}{\dfrac{\partial}{\partial x}Q + p_x\dfrac{\partial Q}{\partial p}}
\end{aligned}
$$

By cross multiplication we have

$$
(\frac{\partial}{\partial x}Q + p_x\frac{\partial Q}{\partial p})p^*_{x^*} = \frac{\partial}{\partial x}\eta + p_x\frac{\partial \eta}{\partial p}
$$

To elaborate on this concept we revisit equation (28), $x'' = -x$. The solutions to this equation were invariant over G_ϵ acting on the function

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \; v : \mathbb{R} \to \mathbb{R}^n \text{ where } y = x'
$$

For a Partial Differential Equation (PDE), we similarly want to extend $G_\epsilon(x, t, p)$ acting on its first prolongation

$$
{G_\epsilon}^{(1)}(x, t, p, p_x, p_t)
$$

or second prolongation

$$
{G_\epsilon}^{(2)}(x, t, p, p_x, p_t, p_{xx}, p_{tt})
$$

or even further, up to any prolongation necessary without becoming intractable. Thus,

$$
\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + P\frac{\partial}{\partial p}
$$

and its first prolongation

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + P\frac{\partial}{\partial p} + \Phi_1\frac{\partial}{\partial p_x} + \Phi_2\frac{\partial}{\partial p_t}$$

Therefore the results change the representation of the group but not the group itself (i.e, $G_\epsilon^{(1)} \equiv G_\epsilon$) as shown in the following theorem. For that purpose, we first introduce the ***Total Derivative Operator***.

Definition 5.3 *[8] Let $F(x, p, p_x, \ldots, p_x^{(n)})$ be a function. Then the **total derivative operator** for x, denoted by D_x, is defined by*

$$\partial x + \sum p_i + \partial p_i$$

where $i \geq 0$.

Total derivative operator could be used to prolong the given transformations. However, it is crucial to find out if the prolonged transformations are invariant under the given one-parameter continuous group. Therefore, we prove that prolonged transformations remain unchanged under a one-parameter continuous group in the following theorem.

Theorem 5.4 *[8] If transformations $x^* = f(x, y, a)$ and $y^* = g(x, y, a))$ form a one-parameter group, then their extensions to derivatives y', y'', $\ldots$, $y^{(n)}$ of any order is again a one-parameter group and is called an **extended transformation group**.*

Proof 1 *[8] It suffices to prove the theorem for the first extension. Let a be a canonical parameter. Then*

$$x^{**} = f(x^*, y^*, b) = f(x, y, a + b)$$
$$y^{**} = g(x^*, y^*, b) = g(x, y, a + b)$$

Therefore

$$y^{**\prime} = g(x, y, y', a) = \frac{D_x(g(x, y, a))}{D_x(f(x, y, a))}$$

To prove the theorem we have to show that

$$y^{**\prime} = g(x^*, y^*, y^{*\prime}, b) = g(x, y, y', a + b)$$

The latter can be obtained by using the chain rule $D_x = D_x(f(x, y, a))D_x^*$ and the following simple calculations. We have

$$y^{**\prime} = g(x^*, y^*, y^{*\prime}, b) = \frac{D_x^*(g(x^*, y^*, b))}{D_x(f(x^*, y^*, b))}$$

No we rewrite the last expression by multiplying its numerator and denominator by $D_x(f(x, y, a))$ and invoking the group property of f and g, as follows:

$$\frac{D_x(f(x, y, a))D_x^*(g(x^*, y^*, b))}{D_x(f(x, y, a))D_x^* f(x^*, y^*, b)} = \frac{D_x(g(x, y, a + b))}{D_x(f(x, y, a + b))} = g(x, y, y', a + b)$$

Thus completing the proof. □

We are interested in a second order PDE which contains p_{xx} in the equation $F(x, t, p, p_t, p_{xx}) = 0$. If we apply the classical method to solve this PDE we either find trivial solutions or the classical method becomes intractable. Recall that it is possible that the classical method produces a trivial solution of this PDE because the actual symmetry group of the PDE solution is a subgroup of the group produced by the classical method. Therefore, we can add a condition to the PDE to reduce the size of the group produced by the classical method and allow possible non-trivial solution(s) to be identified.

In order to derive this additional condition, we first notice that the goal is to find a solution $p = f(x,t)$ of

$$F(x, t, p, p_t, p_{xx}) = 0 \tag{34}$$

that is, p a function of x and t, which is a surface in the xtp coordinate system. By adding a condition we want the surface $p = f(x,t)$ to be invariant under the group with infinitesimal generator $\Gamma^{(2)}$.

Note that $p = f(x,t)$ could be considered as a level surface $u(x,t,p) = k$. If $u(x,t,p) = k$ then

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \lambda} + \frac{\partial u}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

If λ is a group parameter, then

$$X\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial t} + P\frac{\partial u}{\partial p} = 0$$

However, if $p = f(x,t)$, then $u = f(x,t) - p$. So

$$X\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial t} + P(-1) = 0$$

or

$$X\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial t} = P$$

which is known as ***Invariant Surface Condition (ISC)*** [7].

To reiterate, since $p = f(x,t)$ is a form of a solution to equation (34) and the differential form of $p = f(x,t)$ or $f(x,t) - p = 0$ then

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial \lambda} = \frac{\partial p}{\partial \lambda} \tag{35}$$

So if $G_\lambda = e^{\Gamma\lambda}$ then equation (17) becomes

$$X\frac{\partial f}{\partial x} + T\frac{\partial f}{\partial t} = P \tag{36}$$

where

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} \tag{37}$$

6 SOLUTIONS TO FITZHUGH-NAGUMO USING THE NON-CLASSICAL METHOD

We are interested in finding solution to the Fitzhugh Nagumo equation

$$p_t = p_{xx} - p^3 + (a+1)p^2 - ap \tag{38}$$

where p is the dependent variable and x and t are independent variables. Consider the Lie group transformations

$$\begin{aligned} x^* &= X(x,t,p,\epsilon) && (39) \\ t^* &= T(x,t,p,\epsilon) && (40) \\ p^* &= P(x,t,p,\epsilon) && (41) \end{aligned}$$

where ϵ is the group parameter and $\epsilon = 0$ results in the group identity. Expanding the transformations above about $\epsilon = 0$ we have

$$\begin{aligned} x^* &= x + \epsilon X(x,t,p) + O(\epsilon^2) && (42) \\ t^* &= t + \epsilon T(x,t,p) + O(\epsilon^2) && (43) \\ p^* &= p + \epsilon P(x,t,p) + O(\epsilon^2) && (44) \end{aligned}$$

where

$$X(x,t,p) = \left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0}, \quad T(x,t,p) = \left.\frac{dt^*}{d\epsilon}\right|_{\epsilon=0}, \quad P(x,t,p) = \left.\frac{dp^*}{d\epsilon}\right|_{\epsilon=0} \tag{45}$$

are the infinitesimals of the group. We can deduce the global form of the group from the infinitesimal form. Therefore, given $X(x,t,p)$, $T(x,tp)$, and $P(x,t,p)$, we can obtain $x^* = X(x,t,p,\epsilon)$, $t^* = T(x,t,p,\epsilon)$, and $p^* = P(x,t,p,\epsilon)$ by integrating

$$\frac{dx^*}{d\epsilon} = X(x^*,t^*,p^*) \tag{46}$$
$$\frac{dt^*}{d\epsilon} = T(x^*,t^*,p^*) \tag{47}$$
$$\frac{dp^*}{d\epsilon} = P(x^*,t^*,p^*) \tag{48}$$

with initial conditions $x^*|_{\epsilon=0} = x$, $t^*|_{\epsilon=0} = t$, and $p^*|_{\epsilon=0} = p$. Therefore, the problem of finding the invariance group is converted to the problem of determining the infinitesimal generators $X(x,t,p)$, $T(x,t,p)$, and $P(x,t,p)$.

Substituting the variables x, t and p with x^*, t^* and p^* respectively into equation (38), yields to equation (49).

$$p^*_{t^*} = p^*_{x^*x^*} - p^{*3} + (a+1)p^{*2} - ap^* \tag{49}$$

As mentioned earlier, the infinitesimal generators of a group is used to reduce the number of variables of the PDE. In order to find the infinitesimals, we need to extend the group to figure out the derivative transformations. Using the chain rule, from equation (49), we need to find

$$\frac{\partial p^*}{\partial x^*}, \quad \frac{\partial p^*}{\partial t^*}, \quad \frac{\partial p^*}{\partial x^* x^*} \tag{50}$$

Therefore we have,

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial p^*}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial p^*}{\partial t}\frac{\partial t}{\partial x^*} \tag{51}$$

and

$$\frac{\partial p^*}{\partial t^*} = \frac{\partial p^*}{\partial x}\frac{\partial x}{\partial t^*} + \frac{\partial p^*}{\partial t}\frac{\partial t}{\partial t^*} \tag{52}$$

Thus we need to calculate

$$\frac{\partial x}{\partial x^*}, \quad \frac{\partial x}{\partial t^*}, \quad \frac{\partial t}{\partial x^*}, \quad \frac{\partial t}{\partial t^*} \tag{53}$$

It follows that

$$\begin{aligned}
\frac{\partial x}{\partial x^*} &= \frac{\partial}{\partial x^*}\left[x^* - \epsilon X(x,t,p(x,t)) + O(\epsilon^2)\right] && (54)\\
&= 1 - \epsilon\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial p}\frac{\partial p}{\partial x}\right)\frac{\partial x}{\partial x^*} + O(\epsilon^2) && (55)\\
&= 1 - \epsilon\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial p}\frac{\partial p}{\partial x}\right) + O(\epsilon^2) && (56)
\end{aligned}$$

Similarly it could be shown that

$$\begin{aligned}
\frac{\partial t}{\partial x^*} &= -\epsilon\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\right) + O(\epsilon^2) && (57)\\
\frac{\partial x}{\partial t^*} &= -\epsilon\left(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial p}\frac{\partial p}{\partial t}\right) + O(\epsilon^2) && (58)\\
\frac{\partial t}{\partial t^*} &= 1 - \epsilon\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right) + O(\epsilon^2) && (59)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\partial p^*}{\partial x^*} &= \frac{\partial}{\partial x^*}[p(x,t) + \epsilon P(x,t,p(x,t))] + O(\epsilon^2)\\
&= \frac{\partial}{\partial x}\left[p(x,t) + \epsilon P(x,t,p(x,t))\right]\frac{\partial x}{\partial x^*} + \frac{\partial p}{\partial t}\frac{\partial t}{\partial x^*} + O(\epsilon^2)\\
&= \frac{\partial p}{\partial x} + \epsilon\left[\frac{\partial P}{\partial x} + \left(\frac{\partial P}{\partial p} - \frac{\partial X}{\partial x}\right)\frac{\partial p}{\partial x} - \frac{\partial X}{\partial p}\left(\frac{\partial p}{\partial x}\right)^2\right.\\
&\qquad \left. - \frac{\partial T}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\frac{\partial p}{\partial t}\right] + O(\epsilon^2)
\end{aligned}$$

In short,

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial p}{\partial x} + \epsilon\Phi^{[x]} + O(\epsilon^2) \tag{60}$$

where $\Phi^{[x]}$ is the infinitesimal for $\dfrac{\partial p^*}{\partial x^*}$ given by

$$\Phi^{[x]} = \frac{\partial P}{\partial x} + \left(\frac{\partial P}{\partial p} - \frac{\partial X}{\partial x}\right)\frac{\partial p}{\partial x} - \frac{\partial X}{\partial p}\left(\frac{\partial p}{\partial x}\right)^2 - \frac{\partial T}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\frac{\partial p}{\partial t} \tag{61}$$

Similarly we can compute $\dfrac{\partial p^*}{\partial t^*}$ which yields to

$$\frac{\partial p^*}{\partial t^*} = \frac{\partial p}{\partial t} + \epsilon\Phi^{[t]} + O(\epsilon^2) \tag{62}$$

where $\Phi^{[t]}$ is the infinitesimal for $\dfrac{\partial p^*}{\partial t^*}$ and is given by

$$\Phi^{[t]} = \frac{\partial P}{\partial t} + \left(\frac{\partial P}{\partial p} - \frac{\partial T}{\partial t}\right)\frac{\partial p}{\partial t} - \frac{\partial T}{\partial p}\left(\frac{\partial p}{\partial t}\right)^2 - \frac{\partial X}{\partial t}\frac{\partial p}{\partial x} - \frac{\partial X}{\partial p}\frac{\partial p}{\partial x}\frac{\partial p}{\partial t} \tag{63}$$

Using the chain rule it can easily be shown that

$$p^*_{x^*x*} = p_{xx} + \epsilon\Phi^{[xx]} + O(\epsilon^2) \tag{64}$$

where

$$\begin{aligned}
\Phi^{[xx]} \quad = \quad & P_{xx} + (2P_{xp} - X_{xx})\,p_x - T_{xx}p_t + (P_{pp} - 2X_{xp})\,p_x^2 \\
& -2T_{xp}p_xp_t - X_{pp}p_x^3 - T_{pp}p_x^2p_t + (P_p - 2X_x)\,p_{xx} - 2T_xp_{xt} \\
& -3X_pp_xp_{xx} - T_pp_{xx}p_t - 2T_pp_{xt}p_x
\end{aligned}$$

is the infinitesimal for $p^*_{x^*x^*}$ or $\dfrac{\partial p^*}{\partial x^*x^*}$.

Next, we substitute the $p^*_{x^*}$, $p^*_{t^*}$ and $p^*_{x^*x^*}$ into equation (49),

$$\begin{gathered}
p^*_{t^*} = p^*_{x^*x^*} - {p^*}^3 + (a+1){p^*}^2 - ap^* \quad \text{or} \\
p^*_{t^*} - p^*_{x^*x^*} + {p^*}^3 - (a+1){p^*}^2 + ap^* = 0
\end{gathered}$$

From this step on, we utilize Maple software to perform the necessary calculations. A hard copy of the entire Maple code is attached to this paper in Appendix A.

Thus

$$
\begin{aligned}
&p^*_{t^*} - p^*_{x^*x^*} + p^{*3} - (a+1)p^{*2} + ap^* = \qquad (65)\\
&\quad \epsilon^3 P^3 + \left(-aP^2 + 3pP^2 - P^2\right)\epsilon^2 + \Big(P_t - P_{xx} - X_p p_x p_t - X_t p_x - T_p p_t^2\\
&\quad +T_{xx} p_t + X_{pp} p_x^3 + T_p p_{xx} p_t + T_{pp} p_x^2 p_t + 2T_x p_{xt} + 2T_{xp} p_x p_t +\\
&\quad +3X_p p_x p_{xx} + 2T_p p_{xt} p_x + p_t P_p - p_t T_t - 2p_x P_{xp} + p_x X_{xx} - p_x^2 P_{pp}\\
&\quad +2p_x^2 X_{xp} - p_{xx} P_p + 2p_{xx} X_x - 2pP + 3p^2 P + aP - 2apP\Big)\epsilon + ap + p^3\\
&\quad +pt - p^2 - p_{xx} - ap^2 = 0
\end{aligned}
$$

Therefore, from equation (65), the coefficients of the ϵ^0 and ϵ must equal zero and since ϵ is very small, we can ignore the coefficients of ϵ with higher powers.

We substitute the equation (38), $p_t = p_{xx} - p^3 + (a+1)p^2 - ap$, into the coefficient of ϵ^0, $ap + p^3 + pt - p^2 - p_{xx} - ap^2$ which results in

$$(a+1)p^2 - p^2 + ap^2 = 0 \qquad (66)$$

Next, we substitute the *Invariant Surface Condition*, $p_t = P - X * p_x$ into coefficient of ϵ which yields to the equation (67).

$$
\begin{aligned}
&\text{AfterISC}: \qquad (67)\\
&\quad P_t - P_{xx} - X_t p_x + X_{pp} p_x^3 + 2T_x p_{xt} + 3X_p p_x p_{xx} + 2T_p p_{xt} p_x\\
&\quad -2p_x P_{xp} + p_x X_{xx} - p_x^2 P_{pp} + 2p_x^2 X_{xp} - p_{xx} P_p + 2p_{xx} X_x - 2pP\\
&\quad +3p^2 P + aP - 2apP + 2T_p P X p_x - T_p p_{xx} X p_x - T_p P^2 + T_{xx} P\\
&\quad +P_p P - T_t P - T_p X^2 p_x^2 - T_{xx} X p_x - P_p X p_x + T_t X p_x + X_p p_x^2 X\\
&\quad -X_p p_x P + T_p p_{xx} P - T_{pp} p_x^3 X + T_{pp} p_x^2 P - 2T_{xp} p_x^2 X + 2T_{xp} p_x P = 0
\end{aligned}
$$

Then we rearrange equation (67) in terms of p_x, p_x^2 and p_x^3 factors which yields to equation (68).

$$\text{AfterISC in terms of coefficients of } \epsilon : \tag{68}$$

$$(-T_{pp}X + X_{pp})\,p_x^3 + \left(2X_{xp} - T_pX^2 - P_{pp} + X_pX + T_{pp}P - 2T_{xp}X\right)p_x^2$$

$$+\,(3X_pp_{xx} - T_pp_{xx}X - X_t + 2T_pPX + T_tX + 2T_pp_{xt} - 2Pxp + Xxx$$

$$-T_{xx}X - P_pX - X_pP + 2T_{xp}P)\,px + P_t - P_{xx} + aP - 2apP + 2T_xp_{xt}$$

$$-T_tP - T_pP^2 + T_{xx}P + P_pP - 2pP + 3p^2P - p_{xx}P_p + 2p_{xx}X_x$$

$$+T_pp_{xx}P = 0$$

In equation (68) the coefficients of p_x, p_x^2 and p_x^3 must be equal to 0. Therefore,

$$\text{Coefficients of } p_x : \tag{69}$$

$$(3X_pp_{xx} - T_pp_{xx}X - X_t + 2T_pPX + T_tX + 2T_pp_{xt} - 2Pxp + Xxx$$

$$-T_{xx}X - P_pX - X_pP + 2T_{xp}P) = 0$$

Also

$$\text{Coefficients of } p_x^2 : \tag{70}$$

$$\left(2X_{xp} - T_pX^2 - P_{pp} + X_pX + T_{pp}P - 2T_{xp}X\right) = 0$$

and

$$\text{Coefficients of } p_x^2 : \tag{71}$$

$$(-T_{pp}X + X_{pp}) = 0$$

Next we perform the following with Maple

$$\text{AfterISC} - \left(p_x * \text{Coefficient of } p_x + p_x^2 * \text{Coefficient of } p_x^2 + p_x^3 * \text{Coefficient of } p_x^3\right)$$

which, after simplification, results in

$$P_t - P_{xx} + aP - 2apP + 2T_x p_{xt} - T_t P - T_p P^2 + T_{xx} P + P_p P - 2pP \tag{72}$$
$$+3p^2 P - p_{xx} P_p + 2p_{xx} P_p + 2p_{xx} + T_p p_{xx} P = 0$$

Next, we substitute $T_x = 0$, $T_p = 0$, $T_t = 0$, $T_{pp} = 0$, $T_{xx} = 0$, $T_{xp} = 0$, in equations (69), (70), (71), and (72) which yield to the following equations.

$$3X_p p_{xx} - X_t - 2P_{xp} + X_x x - P_p X - X_p P = 0 \tag{73}$$

$$2X_{xp} - P_{pp} + X_p X = 0 \tag{74}$$

$$X_{pp} = 0 \tag{75}$$

$$P_t - P_{xx} + aP - 2apP + P_p P - 2pP + 3p^2 P - p_{xx} P_p + 3p^2 P = 0 \tag{76}$$

In equation (73), $3X_p$ is the coefficient of p_{xx}. Therefore

$$X_p = 0 \tag{77}$$

Also, we rearrange equation (76) in terms of p_{xx} factor which yields to

$$(-P_p + 2X_x)\, p_{xx} + P_t - P_{xx} + aP - 2apP + P_p P - 2pP + 3p^2 P = 0 \tag{78}$$

Therefore,

$$-P_p + 2X_x = 0 \tag{79}$$

Next, we subtract (p_{xx} * coefficient of p_{xx}) from equation (76) which after simplification results in equation (80)

$$P_t - P_{xx} + aP - 2apP + P_p P - 2pP + 3p^2 P = 0 \tag{80}$$

We substitute $P_p = 0$, $X_x = 0$, $P_{px} = 0$ and $X_{xx} = 0$ in equation (80) which results in equation (81)

$$P_t - P_{xx} + aP - 2apP - 2pP + 3p^2 P = 0 \tag{81}$$

Equation (81) could be rearranged as

$$\left(a - 2p + 3p^2 - 2ap\right) P + P_p P + P_t - P_{xx} = 0 \tag{82}$$

Therefore, if $P = \psi(p)$, then $P_{xx} = 0$ and $P_t = 0$ and

$$P_p = -a + 2p - 3p^2 + 2ap \tag{83}$$

It then follows that

$$P = -ap + p^2 - p^3 + ap^2 \tag{84}$$

From the equations (79) and (83) we have

$$\begin{aligned} 2X_x &= P_p &(85)\\ 2X_x &= -\left(a - 2p + 3p^2 - 2ap\right) &(86) \end{aligned}$$

Therefore

$$X = -\left(a - 2p + 3p^2 - 2ap\right) \frac{x}{2} \tag{87}$$

From results (84) and (87) we have

$$\Gamma = \left(-ap + p^2 - p^3 + ap^2\right) \frac{\partial}{\partial P} - \left(a - 2p + 3p^2 - 2ap\right) \frac{x}{2} \frac{\partial}{\partial X} + \frac{\partial}{\partial T} \tag{88}$$

where Γ is an infinitesimal generator.

Next, we use the invariant surface condition in the form

$$\begin{aligned} p_t &= P - Xp_x &(89)\\ &= -ap + p^2 - p^3 + ap^2 + \left(a - 2p + 3p^2 - 2ap\right) \frac{x}{2} &(90) \end{aligned}$$

Substituting for p_t from the original equation yields

$$p_{xx} - p^3 + (a+1)\,p^2 - ap = -p^3 + (a+1)\,p^2 - ap\left(a - 2p + 3p^2 - 2ap\right)\frac{x}{2} \qquad (91)$$

which simplifies to

$$p_{xx} = \left(a - 2p + 3p^2 - 2ap\right)\frac{x}{2} \qquad (92)$$

If we write differentiation by x using prime notation, then we have

$$p'' = \left(a - 2p + 3p^2 - 2ap\right)\frac{x}{2} \qquad (93)$$

which is an ordinary differential equation. Solutions to this ordinary differential equation are also solutions to the original Fitzhugh-Nagumo equation, given that the parameters in solving the ordinary differential equation are dependent on t.

7 CONCLUSION

We have utilized the Non-Classical Method to reduce the given partial differential equation, $p_t = p_{xx} - p^3 + (a+1)p^2 - ap$ to an ordinary differential equation, $p'' = (a - 2p + 3p^2 - 2ap)\,\frac{x}{2}$. Finding solutions to equation (93) could still be difficult analytically, but the resulting ordinary differential equation can be solved numerically using software.

For example, in a similar approach, solutions to

$$u_t = u_{xx} - (u - m_1)(u - m_2)(u - m_3) \tag{94}$$

have been found as follows

$$u(x,t) = \frac{c_1 m_1 \Psi_1 + c_2 m_2 \Psi_2 + c_3 m_3 \Psi_3}{c_1 \Psi_1 + c_2 \Psi_2 + c_3 \Psi_3} \tag{95}$$

where

$$\Psi_j(x,t) = e^{\frac{1}{2}\sqrt{2} m_j x - m_j\left(m_1 + m_2 + m_3 - \frac{3}{2} m_j\right)t} \tag{96}$$

for $j = 1, 2, 3$, where c_1, c_2 and c_3 are constants. In particular, if $m_1 = a$, $m_2 = 1$ and $m_3 = 0$, equation (94) produces the solutions to Fitzhugh Nagumo equation

$$u\,(x,t) = \frac{a k_1 e^{\dfrac{1}{2}\left(\pm\sqrt{2}ax + a^2 t\right)} + k_2 e^{\dfrac{1}{2}\left(\pm\sqrt{2}x + t\right)}}{k_1 e^{\dfrac{1}{2}\left(\pm\sqrt{2}ax + a^2 t\right)} + k_2 e^{\dfrac{1}{2}\left(\pm\sqrt{2}x + t\right)} + k_3 e^{(at)}} \tag{97}$$

Our approach is similar, but features a Maple worksheet, a copy of which could be found in Appendix A. In this worksheet, in steps (1)-(3), we defined and implemented a set of transformations x^* denoted, $xStar$, t^* denoted, $tStar$, and p^* denoted, $pStar$. Then we implemented the infinitesimal generators for $p^*_{t^*}$ denoted $pStar\_tStar$, $p^*_{x^*}$

denoted $pStar\_xStar$, and $p^*_{x^*x^*}$ denoted $pStar\_xStarxStar$ in steps (4)-(6). Next we substituted the transformations $xStar$, $tStar$, and $pStar$ into the desired equation,

$$pStar\_tStar - pStar\_xStarxStar + (pStar)^3 - (a+1)(pStar)^2 + apStar = 0 \quad (98)$$

$pStar\_xStar$, $pStar\_tStar$, and $pStar\_xStarxStar$ introduced an ϵ term into equation (98). So we expanded (98) and collected its terms with respect to coefficients of ϵ^0, ϵ, ϵ^2, and ϵ^3. The coefficients of all the powers of ϵ needs to be zero. Due to linearity and also ϵ being very small, we ignored the coefficients of ϵ^2 and ϵ^3 and set the coefficients of ϵ^0, ϵ equal to zero, in steps (9)-(13). Then, in steps (14) and (15) we substituted the invariant surface condition (ISC) into the coefficient of ϵ. In steps (16)- (19) we collected the coefficients of p_x, p_x^2, and p_x^3 in the coefficient of ϵ and set them equal to zero. Next, we performed the following and simplified the equation to find the remaining of the of the original equation after ISC substitution.

$$\text{AfterISC} - p_x\text{Coeff}p_x - p_x^2\text{Coeff}p_x^2 - p_x^3\text{Coeff}p_x^3 = 0 \quad (99)$$

in steps (20) and (21). Next, we substituted the $T_x = 0$, $T_p = 0$, $T_t = 0$, $T_{pp} = 0$, $T_{xx} = 0$, $T_{pp} = 0$, and $T_{xp} = 0$ into the coefficients of p_x, p_x^2, and p_x^3 which yielded to the equations produced by Maple in step (22). Then, in step (23) we collected the coefficients of p_{xx} in the $AfterISC$ equation and set it equal to zero which yielded to equation (23) and (24). Then, in steps (25) and (26) we implemented the following equation and simplified it and then set it equal to zero.

$$\text{AfterISC} = \text{AfterISC} - p_{xx} * (\text{Coeff}p_{xx}) \quad (100)$$

Then, as the last step, we substituted $P_p = 0$, $X_x = 0$, $P_{px} = 0$, and $X_{xx} = 0$ into the simplified equation produced in step (26) and it resulted in equation (27). Then, we used equation produced in steps (22), (24) and (27) to deduce the infinitesimal generator by find P and X. Then we used the infinitesimal generator to reduce the desired PDE to an ODE.

8 FUTURE RESEARCH

We could apply the method implemented in this Maple worksheet to other equations to find their symmetry solutions. Depending on the infinitesimal generators produced by this method, the reduced ODEs could still be hard to solve. Thus, it is possible to apply this worksheet to the Fitzhugh-Nagumo equation again to find other symmetry solutions for it. In this case, we reduced the Fitzhugh-Nagumo equation to an ODE that could be solved numerically, but in future attempts we might be able to find analytical solutions to this equation.

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APPENDIX A: Nonclassical Method Maple Worksheet

The following is the calculations to *find the coeffecients of like derivatives of* **p.**

```
> restart;
> xStar := x + epsilon * X(x, t, p) + 0*O(epsilon^2);
```

$$xStar := x + \varepsilon\, X(x, t, p) \tag{1}$$

```
> tStar := t + epsilon ; #* #T(x, t, p) + 0*O(epsilon^2);
```

$$tStar := t + \varepsilon \tag{2}$$

```
> pStar := p + epsilon * P(x, t, p)+ 0*O(epsilon^2);
```

$$pStar := p + \varepsilon\, P(x, t, p) \tag{3}$$

```
> pStar_tStar := pt+ epsilon* (
     Pt + (Pp - Tt)*pt
   - Tp * (pt)^2 - Xt* px
   - Xp * px * pt) +  0*O(epsilon^2);
```

$$pStar\_tStar := pt + \varepsilon\, \left(Pt + (Pp - Tt)\, pt - Tp\, pt^2 - Xt\, px - Xp\, px\, pt\right) \tag{4}$$

```
> pStar_xStar:= px+ epsilon* (
     Px + (Pp - Xx)*px
   - Xp * (px)^2 - Tx* pt
   - Tp * px * pt) +  0*O(epsilon^2);
```

$$pStar\_xStar := px + \varepsilon\, \left(Px + (Pp - Xx)\, px - Xp\, px^2 - Tx\, pt - Tp\, px\, pt\right) \tag{5}$$

```
> pStar_xStarxStar := pxx + epsilon*(
     Pxx + (2*Pxp - Xxx)*px
   - Txx*pt
   + (Ppp - 2*Xxp)*(px)^2
   - 2*Txp*px*pt
   - Xpp*(px)^3
   - Tpp*((px)^2)*pt
   + (Pp-2*Xx)*pxx
   - 2*Tx*pxt
   - 3*Xp*px*pxx
   - Tp*pxx*pt
   - 2*Tp*pxx*px
   ) + 0*O(epsilon^2);
```

$$\begin{aligned} pStar\_xStarxStar := {} & pxx + \varepsilon\, \big(Pxx + (2\, Pxp - Xxx)\, px - Txx\, pt + (Ppp - 2\, Xxp)\, px^2 \\ & - 2\, Txp\, px\, pt - Xpp\, px^3 - Tpp\, px^2\, pt + (Pp - 2\, Xx)\, pxx - 2\, Tx\, pxt - 3\, Xp\, px\, pxx \\ & - Tp\, pxx\, pt - 2\, Tp\, pxx\, px\big) \end{aligned} \tag{6}$$

```
> pStar_tStar - pStar_xStarxStar + (pStar)^3 - (a+1)*(pStar)^2 +
  a*pStar;
```

$$pt+\varepsilon\left(Pt+(Pp-Tt)\,pt-Tp\,pt^2-Xt\,px-Xp\,px\,pt\right)-pxx-\varepsilon\left(Pxx+(2\,Pxp-Xxx)\,px-Txx\,pt+(Ppp-2\,Xxp)\,px^2-2\,Txp\,px\,pt-Xpp\,px^3-Tpp\,px^2\,pt+(Pp-2\,Xx)\,pxx-2\,Tx\,pxt-3\,Xp\,px\,pxx-Tp\,pxx\,pt-2\,Tp\,pxx\,px\right)+\left(p+\varepsilon\,P(x,t,p)\right)^3-(a+1)\left(p+\varepsilon\,P(x,t,p)\right)^2+a\left(p+\varepsilon\,P(x,t,p)\right) \quad \mathbf{(7)}$$

```
> subs(P(x,t,p)=P,%);
```

$$pt+\varepsilon\left(Pt+(Pp-Tt)\,pt-Tp\,pt^2-Xt\,px-Xp\,px\,pt\right)-pxx-\varepsilon\left(Pxx+(2\,Pxp-Xxx)\,px-Txx\,pt+(Ppp-2\,Xxp)\,px^2-2\,Txp\,px\,pt-Xpp\,px^3-Tpp\,px^2\,pt+(Pp-2\,Xx)\,pxx-2\,Tx\,pxt-3\,Xp\,px\,pxx-Tp\,pxx\,pt-2\,Tp\,pxx\,px\right)+(p+\varepsilon\,P)^3-(a+1)\,(p+\varepsilon\,P)^2+a\,(p+\varepsilon\,P) \quad \mathbf{(8)}$$

```
> myEqn:=expand(%);
```

$$myEqn:=-\varepsilon\,pt\,Tt+p^3+\varepsilon\,Tp\,pxx\,pt+\varepsilon\,pt\,Pp-2\,\varepsilon\,px\,Pxp+2\,\varepsilon\,px^2\,Xxp+\varepsilon\,px\,Xxx+\varepsilon\,Txx\,pt-2\,p\,\varepsilon\,P-\varepsilon\,Xt\,px-\varepsilon\,Tp\,pt^2+\varepsilon\,Xpp\,px^3+3\,p\,\varepsilon^2\,P^2+2\,\varepsilon\,Tx\,pxt-a\,\varepsilon^2\,P^2-\varepsilon\,px^2\,Ppp+3\,p^2\,\varepsilon\,P-\varepsilon\,pxx\,Pp+2\,\varepsilon\,pxx\,Xx+\varepsilon\,Pt-\varepsilon\,Pxx-p^2-\varepsilon^2\,P^2+\varepsilon^3\,P^3-a\,p^2+a\,p-2\,a\,p\,\varepsilon\,P+2\,\varepsilon\,Tp\,pxx\,px+3\,\varepsilon\,Xp\,px\,pxx+\varepsilon\,Tpp\,px^2\,pt-\varepsilon\,Xp\,px\,pt+2\,\varepsilon\,Txp\,px\,pt+pt-pxx+a\,\varepsilon\,P \quad \mathbf{(9)}$$

```
> collect(myEqn,epsilon);
```

$$\varepsilon^3\,P^3+\left(-a\,P^2+3\,p\,P^2-P^2\right)\varepsilon^2+\left(-Tp\,pt^2-Xt\,px+Txx\,pt-Xp\,px\,pt+2\,Tx\,pxt+Xpp\,px^3+Tpp\,px^2\,pt+Tp\,pxx\,pt+2\,Txp\,px\,pt+3\,Xp\,px\,pxx+2\,Tp\,pxx\,px+pt\,Pp-pt\,Tt-2\,px\,Pxp+px\,Xxx-px^2\,Ppp+2\,px^2\,Xxp-pxx\,Pp+2\,pxx\,Xx-2\,p\,P-2\,a\,p\,P+3\,p^2\,P+a\,P-Pxx+Pt\right)\varepsilon-p^2+p^3+pt-pxx+a\,p-a\,p^2 \quad \mathbf{(10)}$$

```
> epsilon0Coeff := pt-p^2-a*p^2+p^3-pxx+a*p;
  epsilon1Coeff := Pt+Xpp*px^3-Pxx-Tp*pt^2-Xt*px+Txx*pt-Xp*px*pt+
  Tpp*px^2*pt+Tp*pxx*pt+2*Txp*px*pt+3*Xp*px*pxx+2*Tp*pxx*px+2*Tx*
  pxt+pt*Pp-pt*Tt-2*px*Pxp+px*Xxx-px^2*Ppp+2*px^2*Xxp-pxx*Pp+2*
  pxx*Xx-2*p*P+3*p^2*P+a*P-2*a*p*P ;
```

$$epsilon0Coeff:=-p^2+p^3+pt-pxx+a\,p-a\,p^2$$

$$epsilon1Coeff:=-Tp\,pt^2-Xt\,px+Txx\,pt-Xp\,px\,pt+2\,Tx\,pxt+Xpp\,px^3+Tpp\,px^2\,pt+Tp\,pxx\,pt+2\,Txp\,px\,pt+3\,Xp\,px\,pxx+2\,Tp\,pxx\,px+pt\,Pp-pt\,Tt-2\,px\,Pxp+px\,Xxx-px^2\,Ppp+2\,px^2\,Xxp-pxx\,Pp+2\,pxx\,Xx-2\,p\,P-2\,a\,p\,P+3\,p^2\,P+a\,P-Pxx+Pt \quad \mathbf{(11)}$$

```
> subs(pt = pxx - p^3+(a+1)*p^2-a*p, epsilon0Coeff);
```

$$-p^2+(a+1)\,p^2-a\,p^2 \quad \mathbf{(12)}$$

```
> simplify(%);
```

$$0 \quad \mathbf{(13)}$$

```
> AfterISC := subs(pt = P - X*px, epsilon1Coeff);
```

$$AfterISC := -Xt\,px + 2\,Tx\,pxt + Xpp\,px^3 + 3\,Xp\,px\,pxx + 2\,Tp\,pxx\,px - 2\,px\,Pxp + px\,Xxx - px^2\,Ppp + 2\,px^2\,Xxp - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - 2\,a\,p\,P + 3\,p^2\,P + a\,P - (P - X\,px)\,Tt + (P - X\,px)\,Pp + Txx\,(P - X\,px) - Tp\,(P - X\,px)^2 - Xp\,px\,(P - X\,px) + Tpp\,px^2\,(P - X\,px) + Tp\,pxx\,(P - X\,px) + 2\,Txp\,px\,(P - X\,px) - Pxx + Pt \quad \mathbf{(14)}$$

```
> simplify(%);
```

$$-Xt\,px + 2\,Tx\,pxt + Xpp\,px^3 + 3\,Xp\,px\,pxx + 2\,Tp\,pxx\,px - 2\,px\,Pxp + px\,Xxx - px^2\,Ppp + 2\,px^2\,Xxp - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - 2\,a\,p\,P + 2\,Tp\,P\,X\,px - Tp\,pxx\,X\,px + 3\,p^2\,P + a\,P - Tt\,P + Pp\,P + Txx\,P - Tp\,P^2 + Tt\,X\,px - Pp\,X\,px - Txx\,X\,px - Tp\,X^2\,px^2 - Xp\,px\,P + Xp\,px^2\,X + Tp\,pxx\,P + 2\,Txp\,px\,P - 2\,Txp\,px^2\,X - Pxx + Pt + Tpp\,px^2\,P - Tpp\,px^3\,X \quad \mathbf{(15)}$$

```
> collect(%, px);
```

$$(Xpp - Tpp\,X)\,px^3 + \left(-Ppp - Tp\,X^2 + 2\,Xxp + Tpp\,P - 2\,Txp\,X + Xp\,X\right)\,px^2 + (-Xt + Tt\,X + 2\,Txp\,P + 3\,Xp\,pxx + 2\,Tp\,P\,X + Xxx - Xp\,P - Pp\,X - Txx\,X + 2\,Tp\,pxx - Tp\,pxx\,X - 2\,Pxp)\,px - Tt\,P + 2\,Tx\,pxt + 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P + Txx\,P - Tp\,P^2 - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - Pxx + Tp\,pxx\,P + Pt \quad \mathbf{(16)}$$

```
> pxCoeff := -Xt+Tt*X+2*Txp*P+3*Xp*pxx+2*Tp*P*X+Xxx-Xp*P-Pp*X-Txx*
  X+2*Tp*pxx-Tp*pxx*X-2*Pxp;
```

$$pxCoeff := -Xt + Tt\,X + 2\,Txp\,P + 3\,Xp\,pxx + 2\,Tp\,P\,X + Xxx - Xp\,P - Pp\,X - Txx\,X + 2\,Tp\,pxx - Tp\,pxx\,X - 2\,Pxp \quad \mathbf{(17)}$$

```
> px2Coeff := -Ppp-Tp*X^2+2*Xxp+Tpp*P-2*Txp*X+Xp*X;
```

$$px2Coeff := -Ppp - Tp\,X^2 + 2\,Xxp + Tpp\,P - 2\,Txp\,X + Xp\,X \quad \mathbf{(18)}$$

```
> px3Coeff := Xpp-Tpp*X;
```

$$px3Coeff := Xpp - Tpp\,X \quad \mathbf{(19)}$$

```
> AfterISC := AfterISC-pxCoeff*px-px2Coeff*px^2-px3Coeff*px^3;
```

$$AfterISC := -Xt\,px + 2\,Tx\,pxt + Xpp\,px^3 + 3\,Xp\,px\,pxx + 2\,Tp\,pxx\,px - 2\,px\,Pxp + px\,Xxx - px^2\,Ppp + 2\,px^2\,Xxp - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - 2\,a\,p\,P + 3\,p^2\,P + a\,P - (P - X\,px)\,Tt + (P - X\,px)\,Pp + Txx\,(P - X\,px) - Tp\,(P - X\,px)^2 - Xp\,px\,(P - X\,px) + Tpp\,px^2\,(P - X\,px) + Tp\,pxx\,(P - X\,px) + 2\,Txp\,px\,(P - X\,px) - (-Xt + Tt\,X + 2\,Txp\,P + 3\,Xp\,pxx + 2\,Tp\,P\,X + Xxx - Xp\,P - Pp\,X - Txx\,X + 2\,Tp\,pxx - Tp\,pxx\,X - 2\,Pxp)\,px - (Xpp - Tpp\,X)\,px^3 - \left(-Ppp - Tp\,X^2 + 2\,Xxp + Tpp\,P - 2\,Txp\,X + Xp\,X\right)\,px^2 - Pxx + Pt \quad \mathbf{(20)}$$

```
> AfterISC:= expand(%);
```

$$AfterISC := -Tt\,P + 2\,Tx\,pxt + 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P + Txx\,P - Tp\,P^2 - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - Pxx + Tp\,pxx\,P + Pt \quad \mathbf{(21)}$$

```
> pxCoeff:= subs(Tx = 0, Tp = 0, Tt = 0, Tpp = 0, Txx = 0, Tpp =
  0, Txp=0, pxCoeff);
  px2Coeff:= subs(Tx = 0, Tp = 0, Tt = 0, Tpp = 0, Txx = 0, Tpp =
  0, Txp=0, px2Coeff);
  px3Coeff:= subs(Tx = 0, Tp = 0, Tt = 0, Tpp = 0, Txx = 0, Tpp =
  0, Txp=0, px3Coeff);
  AfterISC:= subs(Tx = 0, Tp = 0, Tt = 0, Tpp = 0, Txx = 0, Tpp =
  0, Txp=0, AfterISC);
```

$$pxCoeff := -Xt + 3\,Xp\,pxx + Xxx - Xp\,P - Pp\,X - 2\,Pxp$$

$$px2Coeff := -Ppp + 2\,Xxp + Xp\,X$$

$$px3Coeff := Xpp$$

$$AfterISC := 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - Pxx + Pt \tag{22}$$

```
> collect(AfterISC,pxx);
```

$$(2\,Xx - Pp)\,pxx + 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P + Pt - 2\,p\,P - Pxx \tag{23}$$

```
> pxxCoeff:=2*Xx-Pp;
```

$$pxxCoeff := 2\,Xx - Pp \tag{24}$$

```
> AfterISC:= AfterISC-pxx*pxxCoeff;
```

$$AfterISC := 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P - pxx\,Pp + 2\,pxx\,Xx - 2\,p\,P - Pxx + Pt - (2\,Xx - Pp)\,pxx \tag{25}$$

```
> AfterISC:=simplify(%);
```

$$AfterISC := 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pp\,P + Pt - 2\,p\,P - Pxx \tag{26}$$

```
> AfterISC:=subs(Pp=0, Xx=0, Ppx=0,Xxx=0, AfterISC);
```

$$AfterISC := 3\,p^2\,P + a\,P - 2\,a\,p\,P + Pt - 2\,p\,P - Pxx \tag{27}$$

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